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Brownian Motion Penetrating Fractals

An Application of the Trace Theorem of Besov Spaces

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For a closed connected set F in $\mathbf{R}^n$, assume that there is a local regular Dirichlet form (a symmetric diffusion process) on F whose domain is included in a Lipschitz space or a Besov space on F. Under some condition for the order of the space and the Newtonian 1-capacity of F, we prove that there exists a symmetric diffusion process on $\mathbf{R}^n$ which moves like the process on F and like Brownian motion on $\mathbf{R}^n$ outside F. As an application, we will show that when F is a nested fractal or a Sierpinski carpet whose Hausdorff dimension is greater than $n-2$, we can construct Brownian motion penetrating the fractal. For the proof, we apply the technique developed in the theory of Besov spaces. © 2000 Academic Press

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1. INTRODUCTION

Consider a situation that some media F is located on a Euclidean space $\mathbf{R}^n$. Is there a "natural" diffusion process which moves following the heat transfer of F inside the media and moves like an ordinary Brownian motion outside the media? We will call such a process a penetrating process, if exists. When the heat transfer on F is "similar" to that on the Euclidean space, one might construct the process by some perturbation arguments. But what happens when the behavior of the heat transfer on F is completely different from ordinary Brownian motion, for example when F is a fractal?

The first attempt for this problem is by Lindstrøm [21], which solves the question affirmatively when F is a Sierpinski gasket. More precisely, given a pair of positive functions h_1, h_2 where h_1 is defined and harmonic (w.r.t. the Laplacian on the gasket) inside the gasket and h_2 is defined and harmonic outside the gasket, he constructs a penetrating process whose

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equilibrium measure is determined by h_1 and h_2. He uses nonstandard analysis for the construction and his process is not necessarily symmetric. After that, Kusuoka [18] works on this problem by an analytical approach using Dirichlet forms when F is a nested fractal, a class of finitely ramified fractals whose Laplace operator and Brownian motion is well studied. He makes his attention to the Hölder continuity of the elements of the domain of the form on F and tries to extend the functions to $\mathbf{R}^n$. Unfortunately, he needs several strong conditions including the condition of the Hölder order for his arguments to work, and the condition does not hold for all nested fractals. On the other hand, there is a recent development of the theory of Besov spaces on a closed subset F of $\mathbf{R}^n$ (see [12, 15]; see also [10, 23], etc. for related works). In [12], Besov space on F is characterized as a trace of the Besov space on $\mathbf{R}^n$ to F. Related to the theory, the domains of the Dirichlet forms on nested fractals are characterized by some Lipschitz spaces ([13, 22]).

Motivated by the development, we will answer the original question affirmatively when the domain of the Dirichlet form on F is included in a Lipschitz space or a Besov space on F. I.e., we will construct a local regular Dirichlet form on $\mathbf{R}^n$ whose restriction on F is the Dirichlet form on F and the restriction outside F is the standard quadratic form driven by the Laplacian on $\mathbf{R}^n$. Moreover, we will prove that when the Newtonian 1-capacity of F is positive, the Dirichlet form on $\mathbf{R}^n$ is irreducible, and the corresponding diffusion process penetrates into F.

In Section 2, we will explain our framework and the main theorems for the penetrating process. In Section 3, we will prove key propositions of the main theorems. For the purpose, we apply the extension operator from a Besov space on F to that on $\mathbf{R}^n$ used in the trace theorem of the Besov space. In Section 4, we will explain nested fractals, Sierpinski carpets and the Dirichlet forms on them, and show that the assumption of the domains of the forms mentioned above holds for the cases, by characterizing the domains.

After the manuscript was firstly submitted, we were informed of the paper [14] concerning this topic. There the Dirichlet form corresponding to the penetrating process is constructed for the case of 2-dimensional Sierpinski gasket by a slightly different way. In Theorem 2.7, we will show that his Dirichlet form and ours are the same under a fairly wide framework.

2. FRAMEWORK AND MAIN RESULTS

Let $F \subset \mathbf{R}^n$ be a closed connected set. We assume that there is a Borel measure μ on F such that

$$\mu(B(x,2r)) \leqslant c_{2.1}\mu(B(x,r)) \qquad \forall x \in F, \quad r>0, \tag{2.1}$$

$$c_{2.2} \leqslant \mu(B(x,1)) \leqslant c_{2.3} \qquad \forall x \in F, \tag{2.2}$$

for some $c_{2.1}, c_{2.2}, c_{2.3} > 0$. Here $B(x,r) = \{y \in \mathbf{R}^n : \|x-y\| < r\}$ where $\|\cdot\|$ is an Euclidean norm. (2.1) is often called the doubling condition. Under this assumption, we can easily prove the following.

LEMMA 2.1. *There exist positive constants $c_{2.4}, c_{2.5}$ and $s \leqslant n$ such that the following hold.*

$$\mu(B(x,r)) \geqslant c_{2.4} r^s \qquad \forall x \in F, \quad r \leqslant 1, \tag{2.3}$$

$$\mu(B(x,r)) \leqslant c_{2.5} r^s \qquad \forall x \in F, \quad r \geqslant 1. \tag{2.4}$$

Note that by (2.3), we see that F has a Hausdorff dimension $\leqslant s$.

Let β be a real number such that

$$1 - \frac{n-s}{2} < \beta. \tag{2.5}$$

We assume that there is a local regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $\mathbf{L}^2(F, d\mu)$ so that

$$\mathscr{F} \subset \mathrm{Lip}(\beta, 2, \infty)(F), \tag{2.6}$$

where the *Lipschitz space* $\mathrm{Lip}(\beta, 2, \infty)(F)$ is a set of $f \in \mathbf{L}^2(F, d\mu)$ such that

$$\sup_{v \in \mathbf{N} \cup \{0\}} \alpha^{v(2\beta+s)} \iint_{\|x-y\| < c_0 \alpha^{-v}} |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) < \infty$$

for some $\alpha > 1$, $c_0 > 0$. Note that it is easy to show that different values on the constants c_0 and α give equivalent spaces as long as the former is positive and the latter is greater than 1. Clearly, $\mathrm{Lip}(\beta', 2, \infty)(F) \subset \mathrm{Lip}(\beta, 2, \infty)(F)$ for $\beta \leqslant \beta'$.

Let $\mathscr{D}_0$ be the set of connected components of $\mathbf{R}^n \backslash F$ and $\mathscr{D}$ be a subset of $\mathscr{D}_0$. Define $F_{\mathscr{D}} = F \cup (\bigcup_{D \in \mathscr{D}} D)$ which is obviously a closed subset in $\mathbf{R}^n$. Under the above assumption, we define a bilinear form $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ in $\mathbf{L}^2(F_{\mathscr{D}}, d\tilde{m})$, where $d\tilde{m} = dx + d\mu$, as follows,

$$\mathscr{D}(\tilde{\mathscr{E}}) = \left\{ u \in C_0(F_{\mathscr{D}}) : u|_F \in \mathscr{F}, \sum_{D \in \mathscr{D}} a_D \int_D |\nabla u(x)|^2 \, dx < \infty \right\},$$

$$\tilde{\mathscr{E}}(u,v) = \mathscr{E}(u|_F, v|_F) + \tfrac{1}{2} \sum_{D \in \mathscr{D}} a_D \int_D \nabla u(x) \nabla v(x) \, dx \qquad \forall u, v \in \mathscr{D}(\tilde{\mathscr{E}}).$$

Here $\nabla u(x)$, $x \in D$ is a derivative of u on D in the distribution sense, and $\{a_D\}_{D \in \mathscr{D}}$ is a sequence of positive constants which satisfies $\sup_{D \in \mathscr{D}} a_D < \infty$

(for the proof of Theorem 2.3, we can relax this condition, see Remark 3.3) and $C_0(F_{\mathscr{D}})$ is a space of continuous functions on $F_{\mathscr{D}}$ with compact support. Then the following is obvious.

LEMMA 2.2. 1. *$(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is closable in $\mathbf{L}^2(F_{\mathscr{D}}, d\tilde{m})$.*

2. *$\mathscr{D}(\tilde{\mathscr{E}})$ is an algebra.*

3. *$C_0^\infty(D) \subset \mathscr{D}(\tilde{\mathscr{E}})$ for each $D \in \mathscr{D}$.*

Now, denote $\tilde{\mathscr{F}} = \overline{\mathscr{D}(\tilde{\mathscr{E}})}^{\tilde{\mathscr{E}}_1}$ so that $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is the smallest extension of $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$, where $\tilde{\mathscr{E}}_1(f, f) = \tilde{\mathscr{E}}(f, f) + \|f\|^2_{\mathbf{L}^2}$. Our first main theorem is the following.

THEOREM 2.3. *$(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is a local regular Dirichlet form on $\mathbf{L}^2(F_{\mathscr{D}}, d\tilde{m})$.*

Note that by the general theory ([8]), there is a one to one correspondence between a local regular Dirichlet form on $\mathbf{L}^2(F_{\mathscr{D}}, d\tilde{m})$ and a $\tilde{m}$-symmetric diffusion process on $F_{\mathscr{D}}$ except some exceptional set of starting points. We will denote $\{\tilde{X}_t\}_{t \geqslant 0}$ the diffusion process corresponding to $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$. Note also that when the original form $(\mathscr{E}, \mathscr{F})$ is strong local, then $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is also strong local. The key proposition for the proof of this theorem is the following.

PROPOSITION 2.4. *There is an extension map $\xi\colon \mathscr{F} \cap C_0(F) \to \mathscr{D}(\tilde{\mathscr{E}})$ such that $\xi f|_F = f$.*

We will prove this in the next section. Here we assume this proposition and prove the main theorem.

Proof of Theorem 2.3. It is clear that $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is a local Dirichlet form. For the regularity, it is enough to show that $\mathscr{D}(\tilde{\mathscr{E}})$ is dense in $C_0(F_{\mathscr{D}})$ w.r.t. $\|\cdot\|_\infty$-norm and it is dense in $\tilde{\mathscr{F}}$ w.r.t. $\tilde{\mathscr{E}}_1$-norm. But the latter is clear as $\tilde{\mathscr{F}} = \overline{\mathscr{D}(\tilde{\mathscr{E}})}^{\tilde{\mathscr{E}}_1}$. For the former, we need to show by Stone–Weierstrass' theorem that for each $x, y \in F_{\mathscr{D}}$ with $x \neq y$, there is an $f \in \mathscr{D}(\tilde{\mathscr{E}})$ such that $f(x) \neq f(y)$, since $\mathscr{D}(\tilde{\mathscr{E}})$ is an algebra. If either of x or y is in F^c, this holds by Lemma 2.2.3. For the case $x, y \in F$, there exists $g \in \mathscr{F} \cap C_0(F)$ such that $g(x) \neq g(y)$, as $(\mathscr{E}, \mathscr{F})$ is regular. But then, by Proposition 2.4, $\xi g \in \mathscr{D}(\tilde{\mathscr{E}})$ and $\xi g(x) \neq \xi g(y)$ and the proof is completed. ∎

The Besov space $B^{2,q}_\beta(F)$ is defined as the class of functions f on F such that the following norm is finite:

$$\|f\|^2_{\mathbf{L}^2} + \left\{ \sum_{\nu=0}^{\infty} \left\{ \alpha^{\nu(\beta - n/2)} \left\{ \iint_{\|x-y\| < c_0 \alpha^{-\nu}} \frac{|f(x) - f(y)|^2}{n_\nu(x)\, n_\nu(y)}\, d\mu(x)\, d\mu(y) \right\}^{1/2} \right\}^q \right\}^{1/q},$$

where $n_\nu(x)$ denotes $\mu(B(x, \alpha^{-\nu}))$. Note that when μ satisfies (2.9) and $\beta \notin \mathbf{Z}$, then Lip $(\beta, 2, \infty)(F) = B^{2,\infty}_{\beta+(n-s)/2}(F)$ (p. 125, Proposition 3 in [15]). Our first main theorem still holds when the domain of the original form $(\mathscr{E}, \mathscr{F})$ is included in $B^{2,q}_{\beta}(F)$.

PROPOSITION 2.5. *Assume* (2.1), (2.2) *and the following.*

$$\mu(B(x, r)) \leqslant c_{2.6} r^d \qquad \forall x \in F, \quad r \leqslant 1, \tag{2.7}$$

for some $0 \leqslant d \leqslant s$, $c_{2.6} > 0$. *Assume also that instead of* (2.6) *the following holds*,

$$\mathscr{F} \subset B^{2,q}_{\beta}(F) \tag{2.8}$$

for some $1 \leqslant q \leqslant \infty$ *and* $\beta > 1 + s - d$. *Then, Theorem* 2.3 *holds.*

For the proof, we only need to modify the proof of Proposition 2.4, which we will mention in the next section.

In [14], A. Jonsson constructs the Dirichlet form orresponding to the penetrating process into the 2-dimensional Sierpinski gasket. Our next theorem is that his Dirichlet form and ours coincide under a wider framework. For the purpose, we will consider the case $\mathscr{D} = \mathscr{D}_0$ (thus $F_{\mathscr{D}} = \mathbf{R}^n$) for simplicity and will assume the following in the rest of this section.

Assumption 2.6.

$$c_{2.4} r^s \leqslant \mu(B(x, r)) \leqslant c_{2.6} r^s \qquad \forall x \in F, \quad r \leqslant 1, \tag{2.9}$$

$$0 < \inf_{D \in \mathscr{D}} a_D \leqslant \sup_{D \in \mathscr{D}} a_D < \infty, \tag{2.10}$$

$$m(F) = 0 \qquad \text{where } m \text{ is a Lebesgue measure on } \mathbf{R}^n, \tag{2.11}$$

$$(\mathscr{E}, \mathscr{F}) \text{ is irreducible.} \tag{2.12}$$

Note that (2.9) means both of (2.3) and (2.7) hold with $s = d$. Thus the Hausdorff dimension of F $(\hat{F})$ is s and μ $(\hat{\mu})$ is equivalent to the Hausdorff measure. Let $(\mathscr{E}^{\mathbf{R}^n}, W^{1,2}(\mathbf{R}^n))$ be the usual Dirichlet integral on a Sobolev space, i.e.

$$\mathscr{E}^{\mathbf{R}^n}(u, u) = \tfrac{1}{2} \int_{\mathbf{R}^n} |\nabla u(x)|^2 \, dx \qquad \forall u \in W^{1,2}(\mathbf{R}^n),$$

$$W^{1,2}(\mathbf{R}^n) = \{ f \in \mathbf{L}^2(\mathbf{R}^n, dx) : \nabla f \in (\mathbf{L}^2(\mathbf{R}^n, dx))^n \}.$$

Now, when $s > n - 2$, we define

$$\mathscr{F}' = \{ f \in W^{1,2}(\mathbf{R}^n) : f|_F \in \mathscr{F} \}.$$

Here $f|_F$ is a pointwise restriction of $R(f)$ to F, where

$$R(f)(x) = \lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(t)\, dt,$$

for each $x \in \mathbf{R}^n$ where the limit exists. Note that for $f \in W^{1,2}(\mathbf{R}^n)$, this limit exists quasi-everywhere (w.r.t. $\mathscr{E}^{\mathbf{R}^n}$), and therefore μ-a.e. $x \in F$ as $s > n-2$ (see Corollary 5.1.14 of [1]). Thus the definition makes sense. In [14], it is proved that $(\tilde{\mathscr{E}}, \mathscr{F}')$ with $a_D \equiv 1/2$ is a local regular Dirichlet form for 2-dimensional gasket for $n = 2, 3, 4$. The next theorem includes the fact that this form and our form coinsides.

Theorem 2.7. *Under Assumption* 2.6 *and* $s > n-2$, *it holds that*

$$\mathscr{F}' = \tilde{\mathscr{F}}.$$

Note that by this theorem, (2.10) and (2.11), $\sum_D a_D \int_D |\nabla u|^2\, dx$ and the Dirichlet integral are equivalent, i.e. the ratio of them is bounded from above and below by some positive constants independent of the choice of u. For the proof of the theorem, it is enough to show the following proposition, as $(\tilde{\mathscr{E}}, \mathscr{F}')$ is a closed form (, which can be proved using (6.1) of [14]). Define

$$C = C_0^\infty(F^c) \oplus \{\xi f : f \in \mathscr{F}\},$$

where ξ is an extension map appeared in Proposition 2.4, whose domain can be extended to $\mathscr{F}$ (see the next section for details).

Proposition 2.8. *Assume Assumption* 2.6 *and* $s > n-2$. *Then the following holds.*

1. *C is dense in $\mathscr{F}'$ w.r.t. $\tilde{\mathscr{E}}_1$.*
2. *C is dense in $\tilde{\mathscr{F}}$ w.r.t. $\tilde{\mathscr{E}}_1$.*

The proof will be given in the next section.

Next, we will indicate several properties of the process $\{\tilde{X}_t\}_{t \geqslant 0}$ corresponding to $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$. For each $B \subset \mathbf{R}^n$, define

$$\sigma_B = \inf\{t \geqslant 0 : \tilde{X}_t \in B\}.$$

We then have the following concerning the question whether $\tilde{X}_t$ penetrates into F or not.

THEOREM 2.9. *Assume Assumption* 2.6 *and* $s > n-2$. *Then, for any nearly Borel set* B *with positive* 1-*capacity* (*w.r.t.* $\tilde{\mathscr{E}}$),

$$\tilde{P}^x(\sigma_B < \infty) > 0 \qquad \text{for q.e.} \quad x \in \mathbf{R}^n. \tag{2.13}$$

Especially, when B *is either a subset of* F *whose* 1-*capacity w.r.t.* $\mathscr{E}$ *is positive or a subset of* $\mathbf{R}^n$ *whose* 1-*capacity w.r.t.* $\mathscr{E}^{\mathbf{R}^n}$ *is positive, then* (2.13) *holds.*

The key proposition for the proof is the following. This proposition was obtained by discussions with Dr. M. Hino.

PROPOSITION 2.10. *Under Assumption* 2.6, *the following is equivalent.*

(a) F *is invariant w.r.t.* $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$.

(b) *There exists* F' *which equals to* F $\tilde{m}$-*a.e. such that* $Cap_{\mathbf{R}^n} F' = 0$ *where* $Cap_{\mathbf{R}^n} F'$ *is a* 1-*capacity of* F' *w.r.t.* $\mathscr{E}^{\mathbf{R}^n}$.

(c) $s \leqslant n-2$.

Note that when $s \leqslant n-2$, by this proposition F is invariant and the process $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is not irreducible. We will prove this proposition in the next section. Assuming the proposition, the proof of Theorem 2.9 is easy. Indeed, by Proposition 2.10, we see that F is not invariant w.r.t. $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ when $s > n-2$. Using (2.12) and the structure of $\tilde{\mathscr{E}}$, we see that $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is irreducible. Now, the theorem is a consequence of Theorem 4.6.6 in [8].

Another property of the process $\{\tilde{X}_t\}_{t \geqslant 0}$ is on the decay of the heat semigroup. Let $P_t^{\mathscr{E}}, P_t^{\tilde{\mathscr{E}}}$ $(t > 0)$ be semigroups corresponding to $(\mathscr{E}, \mathscr{F})$, $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ respectively.

THEOREM 2.11. *Assume Assumption* 2.6 *and the following,*

$$\|P_t^{\mathscr{E}}\|_{1 \to \infty} \leqslant c_{2.7} t^{-d_s/2} \qquad \forall t > 0, \tag{2.14}$$

by some $c_{2.7} > 0$ *and* $0 < d_s \leqslant n$. *Then, there exists* $c_{2.8} > 0$ *such that the following holds,*

$$\|P_t^{\tilde{\mathscr{E}}}\|_{1 \to \infty} \leqslant \begin{cases} c_{2.8} t^{-n/2}, & \forall t \in (0, 1], \\ c_{2.8} t^{-d_s/2}, & \forall t \in [1, \infty). \end{cases} \tag{2.15}$$

We will prove this in the next section.

3. PROOF

3.1. *Proof of Proposition* 2.8

In this subsection, we will prove Proposition 2.4. The extension operator in this proposition is the one studied in [12] in order to extend Besov space $B^{p,q}_{\beta}(F)$ into the classical Besov space $\Lambda^{p,q}_{\beta}(\mathbf{R}^n)$ for $1 \leqslant p,\ q \leqslant \infty$, $(n-d)/p < \beta \leqslant 1+(n-s)/p$, which we apply for the case $p=2$, $q=\infty$. We will explain it following [12] and [15]. Note that we will prove Proposition 2.4 for the case $F_{\mathscr{D}} = \mathbf{R}^n$ (i.e. $\mathscr{D} = \mathscr{D}_0$) as the rest of the cases then follow easily. Note also that because any $\alpha > 1$ give equivalent spaces for $\mathrm{Lip}(\beta, 2, \infty)(F)$, we will take $\alpha = 2$ in this section (as it is convenient for the following Whitney decomposition).

We first prepare the Whitney decomposition of the complement of F, which has the following properties. It consists of a collection of closed cubes $\{Q_i\}_{i \in \mathbf{N}}$, with mutually disjoint interiors and sides parallel to the axes so that $F^c = \bigcup_i Q_i$. We assume that the sidelength of the cubes is of the form 2^{-M}, $M \in \mathbf{Z}$. Denote the center of Q_i by x_i, its diameter by l_i and its sidelength by s_i. Then $s_i = l_i/\sqrt{n} \in \{2^{-M} \colon M \in \mathbf{Z}\}$. This decomposition further has the following properties,

$$l_i \leqslant d(Q_i, F) \leqslant 4 l_i, \tag{3.1}$$

$$Q_i \cap Q_j \neq \varnothing \Rightarrow l_i/4 \leqslant l_j \leqslant 4 l_i, \tag{3.2}$$

where $d(Q_i, F)$ is the Euclidean distance from Q_i to F. Let $0 < \varepsilon < 1/4$ and put $Q_i^* = (1+\varepsilon)\, Q_i$. Note that by the above properties of $\{Q_i\}$, each point in F^c is contained in at most $N_0(n)$ (which depends only on the Euclidean dimension) cubes Q_i^* and, $Q_i^* \cap Q_j \neq \varnothing$ if and only if $Q_i \cap Q_j \neq \varnothing$. To this decomposition, we associate a partition of unity, consisting of non-negative functions $\{\varphi_i\}_{i \in \mathbf{N}}$ such that $\varphi_i|_{(Q_i^*)^c} = 0$, $\sum_i \varphi_i(x) = 1$ $\forall x \in F^c$, and

$$|D^k \varphi_i(x)| \leqslant A_k (l_i)^{-|k|} \qquad \forall x \in \mathbf{R}^n, \quad i \in \mathbf{N}, \quad k \in (\mathbf{N} \cup \{0\})^n, \tag{3.3}$$

for some constant $A_k > 0$ depending only on k. Here, for $k = (k_1, ..., k_n)$, we set $D^k = (\partial^{k_1}/\partial x_1^{k_1}) \cdots (\partial^{k_n}/\partial x_1^{k_n})$ and $|k| = k_1 + \cdots k_n$. For the moment, we only need (3.3) for $|k| = 0, 1$.

We are now ready to define the extension operator ξ. Set $m_i = \mu(B(x_i, 6 l_i))^{-1}$. Note that when $l_i = \sqrt{n}\, 2^{-\nu}$ for $\nu \in \mathbf{N}$, we have by (2.3) that

$$m_i \leqslant c_{3.1} 2^{\nu s}, \tag{3.4}$$

where $c_{3.1}$ only depends on n. Now, for $f \in \mathbf{L}^2(F, d\mu)$, define

$$\xi f(x) = \sum_{i \in I} \varphi_i(x)\, m_i \int_{\|t - x_i\| \leqslant 6l_i} f(t)\, d\mu(t) \qquad \forall x \in F^c,$$

where $I \equiv \{i \in \mathbf{N} : s_i \leqslant 1\}$. The concrete value 6 is not important; it is enough to choose sufficiently large number α_0 so that $\mu(\{t \colon \|t - x_i\| \leqslant \alpha_0 l_i\} \cap F)$ is bounded away from 0. For each fixed $x \in F^c$, there are only finite number of φ_i where $\varphi_i(x) \neq 0$ so that ξf is well defined and $C^\infty(F^c)$. Further, by (3.2) and by the definition of I, $\xi f(x) = 0$ if $x \in Q_j$, $s_j > 4$. Thus, if $f \in C_0(F)$, then $\xi f \in C_b^\infty(F^c)$ where $C_b^\infty(F^c)$ is a space of infinitely differentiable bounded supported functions on F^c. Before giving more properties of ξf, we note the following,

$$Q_i \cap Q_j \neq \varnothing, \qquad \|t - x_j\| \leqslant 6l_j \Rightarrow \|t - x_i\| \leqslant 30 l_i, \tag{3.5}$$

$$Q_i \cap Q_j \neq \varnothing \Rightarrow m_i \leqslant c_{3.2} m_j, \tag{3.6}$$

for each $i, j \in \mathbf{N}$ where $c_{3.2}$ is a positive constant. The former is because

$$\|t - x_i\| \leqslant \|t - x_j\| + \|x_j - x_i\| \leqslant 6l_j + l_i + l_j \leqslant 30 l_i,$$

where we use (3.2) in the last inequality. Using this, the latter assertion comes by using (2.1).

We next introduce two lemmas from [15] (they are also used in [12]). For the proof, we refer readers to [15] (in fact they are easy exercises). Define

$$J(x_i, x_j) = \left\{ m_i m_j \iint_{\substack{\|t - x_i\| \leqslant 30 l_i \\ \|s - x_j\| \leqslant 30 l_j}} |f(t) - f(s)|^2\, d\mu(t)\, d\mu(s) \right\}^{1/2}.$$

LEMMA 3.1. *Let $x \in Q_i$, $y \in Q_j$ so that $s_i, s_j \leqslant 1/4$. Then there exist positive constants $c_{3.3} \sim c_{3.6}$ ($c_{3.4}$ and $c_{3.6}$ depend on k) such that the following hold.*

1. $|\xi f(x) - \xi f(y)| \leqslant c_{3.3} J(x_i, x_j)$.
2. $|D^k(\xi f)(x)| \leqslant c_{3.4} l_i^{-|k|} J(x_i, x_j)\ \forall\ |k| > 0,\ \forall j \in \mathbf{N}$.
3. $|\xi f(x) - b| \leqslant c_{3.5} \{m_i \int_{\|t - x_i\| \leqslant 30 l_i} |f(t) - b|^2\, d\mu(t)\}^{1/2}\ \forall b \in \mathbf{R}$.
4. $|D^k(\xi f)(x)| \leqslant c_{3.6} l_i^{-|k|} \{m_i \int_{\|t - x_i\| \leqslant 30 l_i} |f(t)|^2\, d\mu(t)\}^{1/2}\ \forall\ |k| \geqslant 0$.

Using this lemma, we can show that if $f \in C_0(F)$, then ξf ($\in C_b^\infty(F^c)$) is uniformly continuous on F^c and therefore can be extended continuously to ∂F. Indeed, it is enough to show the uniform continuity on $F_{\delta'} \equiv \{y \in F^c : d(y, F) \leqslant \delta'\}$ for small $\delta' > 0$. By the uniform continuity of f, we see that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $t, s \in F$ and $\|t - s\| < \delta$, then $|f(t) - f(s)| < \varepsilon$. Now, take $x, y \in F_{\delta'}$, $\|x - y\| \leqslant \delta/2$ and assume $x \in Q_i$, $y \in Q_j$. By taking δ' small enough, we can take $30l_i \vee 30l_j < \delta/6$ (due to (3.1)). Then, $\|x_i - x_j\| \leqslant l_i + l_j + \|x - y\| < 2\delta/3$. Thus, for $t, s \in F$ with $\|t - x_i\| \leqslant 30l_i$, $\|s - x_j\| \leqslant 30l_j$, we have $\|t - s\| \leqslant \|t - x_i\| + \|x_i - x_j\| + \|s - x_j\| < \delta$ so that $|f(t) - f(s)| < \varepsilon$. Applying this to the above lemma (1),

$$|\xi f(x) - \xi f(y)|^2 \leqslant c_{3.3}^2 \varepsilon^2 \frac{\mu(B(x_i, 30l_i))}{\mu(B(x_i, 6l_i))} \frac{\mu(B(x_j, 30l_j))}{\mu(B(x_j, 6l_j))} \leqslant c' \varepsilon^2,$$

where we use (2.1) in the second inequality, and the uniform continuity is proved. Thus, we will regard $\xi f \in C_0(\mathbf{R}^n \setminus \operatorname{Int} F)$ in the following. Let Δ_i denote the union of those cubes Q_i which have sides of length $s_i = 2^{-i}$. The second lemma from [15] is the following.

Lemma 3.2. *Let $a > 0$, and let h be a non-negative function defined on F. Let g be given by*

$$g(x) = \int_{\|t - x_i\| \leqslant a l_i} h(t)\, d\mu(t) \qquad \forall x \in \mathit{Int}\ Q_i, \quad s_i = 2^{-\nu}.$$

Then, for $x_0 \in \mathbf{R}^n$ and $0 < r \leqslant \infty$,

$$\int_{\substack{x \in \Delta_\nu \\ \|x - x_0\| \leqslant r}} g(x)\, dx \leqslant c_{3.7} 2^{-\nu n} \int_{\|t - x_0\| \leqslant r + c_{3.8} 2^{-\nu}} h(t)\, d\mu(t).$$

In particular for $r = \infty$,

$$\int_{x \in \Delta_\nu} g(x)\, dx \leqslant c_{3.7} 2^{-\nu n} \int h(t)\, d\mu(t).$$

Here the constants $c_{3.7}$, $c_{3.8}$ depend only on a and n.

Define $\xi f(x) = f(x)$ for $x \in \operatorname{Int} F$. In order to complete the proof of Proposition 2.4, it is enough to prove that $\sum_{D \in \mathscr{D}} a_D \int_D |\nabla(\xi f)(x)|^2\, dx < \infty$ and $\xi f|_{\partial F} = f$. Note that by (2.6) we have for each $f \in \mathscr{F}$ and for each $\nu \geqslant 0$,

$$\iint_{\|x - y\| < c_0 2^{-\nu}} |f(x) - f(y)|^2\, d\mu(x)\, d\mu(y) < M 2^{-\nu(2\beta + s)}, \tag{3.7}$$

where $M>0$ is a constant depending only on f and not on ν. For the case of (2.8), for each $f\in\mathscr{F}$ and for each $\nu\geqslant 0$,

$$\iint_{\|x-y\|<c_0 2^{-\nu}} \frac{|f(x)-f(y)|^2}{n_\nu(x)\,n_\nu(y)}\,d\mu(x)\,d\mu(y)<M2^{-\nu(2\beta-n)}.$$

As $n_\nu(x)\leqslant c_{2.6}2^{-\nu d}$ $\forall x\in F$ (due to (2.7)), we have

$$\iint_{\|x-y\|<c_0 2^{-\nu}} |f(x)-f(y)|^2\,d\mu(x)\,d\mu(y)<M'2^{-\nu(2\beta-n+2d)}. \tag{3.8}$$

Proof of $\sum_{D\in\mathscr{D}} a_D \int_D |\nabla(\xi f)(x)|^2\,dx<\infty$. Noting that $\xi f(x)=0$ if $x\in\Delta_\nu$, $\nu\leqslant -3$, what we need to show is the following:

$$\sum_{\nu=-2}^{\infty}\int_{\Delta_\nu} a_{\Delta_\nu}\sum_{|j|=1}|D^j(\xi f)(x)|^2\,dx<\infty, \tag{3.9}$$

where $a_{\Delta_\nu}\equiv\sup_{Q_i\subset D,\,\exists Q_i\in\Delta_\nu} a_D$.

Using Lemma 3.1(2) with $i=j$ and (3.4), we have for $x\in Q_i\subset\Delta_\nu$, $\nu\geqslant 2$ and $|j|=1$,

$$|D^j(\xi f)(x)|^2\leqslant c2^{2\nu(1+s)}\iint_{\substack{\|t-x_i\|\leqslant 30l_i\\ \|t-s\|\leqslant c'2^{-\nu}}} |f(t)-f(s)|^2\,d\mu(t)\,d\mu(s),$$

where we use the fact that if $\|s-x_i\|\leqslant 30l_i$, $\|t-x_i\|\leqslant 30l_i$ then $\|s-t\|\leqslant 60l_i\leqslant c'2^{-\nu}$ for some c' which depends only on n. Using Lemma 3.2 with $a=30$, $r=\infty$ and the integral w.r.t. s above as $h(t)$, we obtain

$$\begin{aligned}
&\sum_{\nu=2}^{\infty} a_{\Delta_\nu}\int_{\Delta_\nu}\sum_{|j|=1}|D^j(\xi f)(x)|^2\,dx\\
&\qquad\leqslant c''\sum_{\nu=2}^{\infty} a_{\Delta_\nu}2^{\nu(2+2s-n)}\iint_{\|t-s\|\leqslant c'2^{-\nu}}|f(t)-f(s)|^2\,d\mu(t)\,d\mu(s)\\
&\qquad\leqslant c''M\sum_\nu a_{\Delta_\nu}2^{\nu(2(1-\beta)+s-n)}<\infty,
\end{aligned}\tag{3.10}$$

where we use (3.7), (2.5) and the fact $\sup_\nu a_{\Delta_\nu}<\infty$ in the last inequality. For the case of (3.8), the last term is $c''M\sum_\nu a_{\Delta_\nu}2^{2\nu(1-\beta+s-d)}$ which is finite when $\beta>1+s-d$.

For $-2\leqslant\nu\leqslant 2$, using Lemma 3.1(4) and Lemma 3.2 with $r=\infty$, we have

$$\int_{\Delta_\nu}\sum_{|j|=1}|D^j(\xi f)(x)|^2\,dx\leqslant c\,\|f\|_{\mathbf{L}^2}$$

so that (3.9) is proved. ∎

Proof of $\xi f|_{\partial F} = f$. As $\xi f \in C_0(\mathbf{R}^n \backslash \operatorname{Int} F)$ and $f \in C_0(F)$, it is enough to show that for each $x_0 \in \partial F$,

$$\lim_{\substack{x \to x_0 \\ x \in F^c}} \xi f(x) = f(x_0). \tag{3.11}$$

For small $\delta > 0$ and $x_0 \in \partial F$, take $x \in F^c$ such that $\|x - x_0\| < \delta$. If $\varphi_i(x) \neq 0$, then $x \in Q_i^*$ so that $\|x - x_i\| \leqslant (1 + \varepsilon)\, l_i/2 \leqslant l_i$. By (3.1), $(1 - \varepsilon)\, l_i \leqslant d(Q_i, F) \leqslant \|x - x_0\| < \delta$. Using these facts, we have for $\|t - x_i\| \leqslant 6l_i$,

$$\begin{aligned} \|t - x_0\| &\leqslant \|t - x_i\| + \|x_i - x\| + \|x - x_0\| \\ &\leqslant 6l_i + l_i + \delta \leqslant \left(\frac{7}{1 - \varepsilon} + 1\right) \delta \leqslant 11\delta. \end{aligned}$$

Now, as f is uniformly continuous, $\forall \varepsilon, \exists \delta'$ such that if $t, s \in F$ and $\|t - s\| < \delta'$, then $|f(t) - f(s)| < \varepsilon$. Take $\delta < \delta'/11$. By the above fact, if $\varphi_i(x) \neq 0$, we have

$$m_i \int_{\|t - x_i\| \leqslant 6l_i} |f(t) - f(x_0)|\, d\mu(t) \leqslant m_i \mu(B(x_i, 6l_i))\varepsilon = \varepsilon.$$

Thus,

$$\begin{aligned} |\xi f(x) - f(x_0)| &\leqslant \sum_{i \in I} \varphi_i(x)\, m_i \int_{\|t - x_i\| \leqslant 6l_i} |f(t) - f(x_0)|\, d\mu(t) \\ &\leqslant \varepsilon \sum_i \varphi_i(x) \leqslant \varepsilon, \end{aligned}$$

so that (3.11) is proved. ∎

Remark 3.3. By (3.10), we see that we can relax the condition $\sup_D a_D < \infty$ to the following:

$$\sum_\nu a_{\Delta_\nu} 2^{\nu(2(1 - \beta) + s - n)} < \infty. \tag{3.12}$$

3.2. *Proof of Proposition* 2.8, 2.10, *and Theorem* 2.11

Proof of Proposition 2.8. We note that (1) can be proved by extending the argument in [14] Section 5, 6 to our framework, which is a good exercise. So here we only prove (2). First, we will show

$$C' = C_0^\infty(F^c) \oplus \{\xi f \colon f \in \mathscr{F} \cap C_0(F)\} \subset C,$$

is dense in $\tilde{\mathscr{F}}$ (w.r.t. $\tilde{\mathscr{E}}_1$). By Lemma 2.2(3) and Proposition 2.4, we know that $C' \subset \mathscr{D}(\tilde{\mathscr{E}})$. Thus, it is enough to show that C' is dense in $\mathscr{D}(\tilde{\mathscr{E}})$. take $f \in \mathscr{D}(\tilde{\mathscr{E}})$ arbitrary. Then, by (2.10) and Proposition 2.4, we see that $f - \xi(f|_F) \in W^{1,2}(F^c) \cap C_0(F^c)$. Thus, there is a sequence $\{f_n\}_n \subset C_0^\infty(F^c)$ such that $f_n \to f - \xi(f|_F)$ in $W^{1,2}(F^c)$. As $f - \xi(f|_F) = 0$ in F, we have $\{f_n + \xi(f|_F)\} \subset C'$ and $f_n + \xi(f|_F) \to f - \xi(f|_F) + \xi(f|_F) = f$ in $\tilde{\mathscr{E}}_1$ which shows that C' is dense in $\mathscr{D}(\tilde{\mathscr{E}})$. Now the proof is completed once we show

$$\{\xi f : f \in \mathscr{F}\} \subset \tilde{\mathscr{F}}. \tag{3.13}$$

But under Assumption 2.6 and $s > n-2$, ξ can be extended to a continuous operator from $\mathrm{Lip}(\beta, 2, \infty)(F)$ to $B^{2,\infty}_{\beta+(n-s)/2}(\mathbf{R}^n)$ with $R(\xi f) = f\mu$-a.e. (see, for example, Corollary 1 in [13]). On the other hand, by (2.5) and the Sobolev type imbedding theorem, $B^{2,\infty}_{\beta+(n-s)/2}(\mathbf{R}^n) \subset W^{1,2}(\mathbf{R}^n)$. Using the facts, (3.13) is easily proved. ∎

Proof of Proposition 2.10. Note that by Theorem 1.6.1 of [8], (a) is equivalent to the following two conditions,

$$f \in \tilde{\mathscr{F}} \Rightarrow 1_F f \in \tilde{\mathscr{F}}, \tag{3.14}$$

$$u, v \in \tilde{\mathscr{F}} \Rightarrow \tilde{\mathscr{E}}(u, v) = \tilde{\mathscr{E}}(1_F u, 1_F v) + \tilde{\mathscr{E}}(1_{\mathbf{R}^n \setminus F} u, 1_{\mathbf{R}^n \setminus F} v). \tag{3.15}$$

For (b) $\Rightarrow$ (a), we first prove (3.14) and (3.15) for $f, u, v \in \mathscr{D}(\tilde{\mathscr{E}})$. Let $K = \mathrm{Supp}\, f$ which is compact. From (b), there exists a sequence of open sets O_l which contains $F' \cap K$ and $\mathrm{Cap}_{\mathbf{R}^n} O_l \leqslant 1/l$. Thus, for each l, there exist $\{g_m^{(l)}\}_m \subset W^{1,2}(\mathbf{R}^n) \cap C_0(\mathbf{R}^n)$ such that $0 \leqslant g_m^{(l)} \leqslant 1$, $g_m^{(l)}|_{O_l} = 1$ and $\lim_m \mathscr{E}_1^{\mathbf{R}^n}(g_m^{(l)}, g_m^{(l)}) = \mathrm{Cap}_{\mathbf{R}^n} O_l$. We can thus take $g_l \in W^{1,2}(\mathbf{R}^n) \cap C_0(\mathbf{R}^n)$ so that $\mathscr{E}_1^{\mathbf{R}^n}(g_l, g_l) \leqslant 2/l$ for all l. By taking a subsequence, if necessary, $g_l \to 0$ m-a.e.. Thus, $fg_l \to f1_F$ $\tilde{m}$-a.e.. Noting that $fg_l \in \mathscr{D}(\tilde{\mathscr{E}})$ and $fg_l|_{F'} = f$, we have

$$\begin{aligned}\tilde{\mathscr{E}}_1(fg_l, fg_l) &\leqslant c_1 \mathscr{E}_1^{\mathbf{R}^n}(fg_l, fg_l) + \mathscr{E}_1^F(f, f)\\ &\leqslant 2c_1 \|f\|_\infty^2 \mathscr{E}_1^{\mathbf{R}^n}(g_l, g_l) + 2c_1 \mathscr{E}^{\mathbf{R}^n}(f, f) + \mathscr{E}_1^F(f|_F, f|_F),\end{aligned}$$

where $c_1 = \sup_D a_D < \infty$. As the right hand side is uniformly bounded w.r.t. l, we have by Banach–Alaoglu and Banach–Saks theorems that there exists a subsequence $\{g_{n_k}\}_k$ such that $fg_l' \equiv f(1/N) \sum_{k=1}^N g_{n_k} \to f1_F$ in $\tilde{\mathscr{E}}_1$ as $l \to \infty$, which proves (3.14). For (3.15), it is enough to prove $\tilde{\mathscr{E}}(1_F u, 1_{\mathbf{R}^n \setminus F} v) = 0$. Take $\{u_n\}, \{v_n\} \subset \mathscr{D}(\tilde{\mathscr{E}})$ such that $u_n \to 1_F u$ in $\tilde{\mathscr{E}}_1$ and $\tilde{m}$-a.e., $v_n \to 1_{\mathbf{R}^n \setminus F} v$ in $\tilde{\mathscr{E}}_1$ and $\tilde{m}$-a.e.. Then, $u_n \to 0$ in $\mathscr{E}_1^{\mathbf{R}^n}$, because $\{u_n\}$ is $\mathscr{E}_1^{\mathbf{R}^n}$-Cauchy and $u_n \to 0$ m-a.e.. By the same reason, $v_n|_F \to 0$ in $\mathscr{E}_1^F$. Thus,

$$\tilde{\mathscr{E}}(1_F u, 1_{\mathbf{R}^n \setminus F} v) = \lim_{n \to \infty} \{\mathscr{E}^{\mathbf{R}^n}(u_n, v_n) + \mathscr{E}^F(u_n|_F, v_n|_F)\} = 0,$$

and we obtain the result. Now, by approximation, we can easily prove (3.14) and (3.15) for $f, u, v \in \mathscr{F}$.

We next show (a) $\Rightarrow$ (b). As $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ is a local regular Dirichlet form on $\mathbf{R}^n$, for each $l \in \mathbf{N}$, we have $g_l \in \tilde{\mathscr{F}} \cap C_0(\mathbf{R}^n)$ such that $g_l|_{B(0, \alpha^l)} = 1$, $g_l|_{B(0, \alpha^{l+1})^c} = 0$ where $\alpha > 1$ and $B(0, r) = \{x \in \mathbf{R}^n : \|x\| < r\}$. Define $F_l = F \cap B(0, \alpha^l)$. Then, by (3.14), we have $h_l \equiv 1_F g_l \in \tilde{\mathscr{F}}$. Note that $h_l|_{F^c} = 0$, $h_l|_{F_l} = 1$ and $h_l|_{F^c_{l+1}} = 0$. Let $\tilde{h}_l$ be a quasi-continuous modification of h_l and set $F'_l = \{x \in \mathbf{R}^n : \tilde{h}_l(x) \neq 0\}$. As $\tilde{h}_l = h_l$ $\tilde{m}$-a.e., $F_l \subset F'_l \subset F_{l+1}$ $\tilde{m}$-a.e., especially, $m(F'_l) = 0$ (due to (2.11)). Further, F'_l is quasi-open. Therefore $\mathrm{Cap}_{\mathbf{R}^n}(F'_l) = 0$. Taking $F' = \bigcup_l F'_l$, we have $F = F'$ $\tilde{m}$-a.e. and $\mathrm{Cap}_{\mathbf{R}^n}(F') = 0$.

Finally, we will show (b) $\Leftrightarrow$ (c). It is known that when F satisfies (2.10), (c) is equivalent to $\mathrm{Cap}_{\mathbf{R}^n} F = 0$ (see, for example, remark after Lemma 5.1 in [14]). Thus (c) $\Rightarrow$ (b) holds. On the other hand, assume $s > n-2$ and let F' equals to F $\tilde{m}$-a.e.. Then $F' \cap F$ also satisfies (2.9), thus the Hausdorff dimension of $F' \cap F$ is s. Using the above fact again, we obtain $\mathrm{Cap}_{\mathbf{R}^n}(F' \cap F) > 0$ so that $\mathrm{Cap}_{\mathbf{R}^n} F' > 0$. This proves (b) $\Rightarrow$ (c). ∎

As the proof suggests, the equivalence between (a) and (b) holds under milder condition for μ.

Proof of Theorem 2.11. By (2.14) and by the property of Brownian motion on $\mathbf{R}^n$, we can apply [7] Theorem (2.1) and have the following,

$$\|f\|^{2+4/d_s}_{\mathbf{L}^2(F, d\mu)} \leqslant c_1 \mathscr{E}(f, f)\, \|f\|^{4/d_s}_{\mathbf{L}^1(F, d\mu)} \qquad \forall f \in \mathbf{L}^2(F, d\mu),$$
$$\|f\|^{2+4/n}_{\mathbf{L}^2(\mathbf{R}^n, dx)} \leqslant c_2 \mathscr{E}^{\mathbf{R}^n}(f, f)\, \|f\|^{4/n}_{\mathbf{L}^1(\mathbf{R}^n, dx)} \qquad \forall f \in \mathbf{L}^2(\mathbf{R}^n, dx).$$

We thus obtain

$$\|f\|^2_{\mathbf{L}^2} \leqslant c_1 \mathscr{E}(f, f)^{d_s/(d_s+2)}\, \|f\|^{4/(d_s+2)}_{\mathbf{L}^1} \leqslant c_1 \left(\frac{\mathscr{E}(f, f)}{\|f\|^2_{\mathbf{L}^1}} \right)^{d_s/(d_s+2)} \|f\|^2_{\mathbf{L}^1},$$

for all $f \in \mathbf{L}^2(F, d\mu)$ and a similar estimate holds for $\mathscr{E}^{\mathbf{R}^n}$. Therefore, for each $g \in \mathbf{L}^2(F, d\tilde{m})$,

$$\begin{aligned}
&\|g\|^2_{\mathbf{L}^2(\mathbf{R}^n, d\tilde{m})} \\
&\quad \leqslant 4(\|g\|^2_{\mathbf{L}^2(F, d\mu)} \vee \|g\|^2_{\mathbf{L}^2(\mathbf{R}^n, dx)}) \\
&\quad \leqslant 4(c_1 \mathscr{E}(g, g)^{d_s/(d_s+2)}\, \|g\|^{4/(d_s+2)}_{\mathbf{L}^1(F, d\mu)} \vee c_2 \mathscr{E}^{\mathbf{R}^n}(g, g)^{n/(n+2)}\, \|g\|^{4/(n+2)}_{\mathbf{L}^1(\mathbf{R}^n, dx)}) \\
&\quad \leqslant 4(c_1 \tilde{\mathscr{E}}(g, g)^{d_s/(d_s+2)}\, \|g\|^{4/(d_s+2)}_{\mathbf{L}^1(\mathbf{R}^n, d\tilde{m})} \vee \frac{c_2}{M^{n/(n+2)}} \tilde{\mathscr{E}}(g, g)^{n/(n+2)}\, \|g\|^{4/(n+2)}_{\mathbf{L}^1(\mathbf{R}^n, d\tilde{m})}) \\
&\quad \leqslant c_3 \left\{ \left(\frac{\tilde{\mathscr{E}}(g, g)}{\|g\|^2_{\mathbf{L}^1(\mathbf{R}^n, d\tilde{m})}} \right)^{d_s/(d_s+2)} + \left(\frac{\tilde{\mathscr{E}}(g, g)}{\|g\|^2_{\mathbf{L}^1(\mathbf{R}^n, d\tilde{m})}} \right)^{n/(n+2)} \right\} \|g\|^2_{\mathbf{L}^1(\mathbf{R}^n, d\tilde{m})} \cdot
\end{aligned}$$

Here $M = \inf_D a_D > 0$ and we apply the above facts in the second and the last inequality. We thus obtain the result using Corollary (2.12) in [7]. ▮

4. DIRICHLET FORMS ON FRACTALS AND THEIR DOMAINS

In this section, we will consider nested fractals and Sierpinski carpets and show that the Dirichlet forms on them satisfy (2.6).

4.1. *Nested fractals and Sierpinski carpets*

For $\alpha > 1$, let $\{\Psi_i\}_{i=1}^N$ be *α-similitude maps* on $\mathbf{R}^n$, i.e. $\Psi_i \mathbf{x} = \alpha^{-1} U_i \mathbf{x} + \gamma_i$, $\mathbf{x} \in \mathbf{R}^n$ for some unitary maps U_i, $\gamma_i \in \mathbf{R}^n$. We assume without loss of generality that $\Psi_1(\mathbf{x}) = \alpha^{-1}\mathbf{x}$. We also assume the *open set condition* for $\{\Psi_i\}_{i=1}^N$, i.e. there is a non-empty, bounded open set W such that $\{\Psi_i(W)\}_i$ are disjoint and $\bigcup_{i=1}^N \Psi_i(W) \subset W$. As $\{\Psi_i\}_{i=1}^N$ is a family of contraction maps, there exists a unique non-void compact set $\hat{F}$ such that $\hat{F} = \bigcup_{i=1}^N \Psi_i(\hat{F})$. We will consider the case $\hat{F}$ is connected. Denote $\Psi_{i_1, ..., i_n} = \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}$ and $S = \{1, 2, ..., N\}$. We will call the set $\Psi_{i_1, ..., i_n}(\hat{F})$ an n-complex. We sometimes identify an n-complex and the corresponding element of S^n. Define $F = \bigcup_{n=1}^\infty \alpha^n \hat{F}$. We now give the definition of nested fractals and Sierpinski carpets following [20] and [5].

1. *Nested fractals.* Let V be the set of fixed points of the Ψ_i's, $i \in S$. A point $x \in V$ is called an essential fixed point if there exist $i, j \in S$, $i \neq j$ and $y \in V$ such that $\Psi_i(x) = \Psi_j(y)$. We write V_0 for the set of essential fixed points. Define F_0 the convex hull of V_0. Set $V_n = \bigcup_{i_1, ..., i_n = 1}^N \Psi_{i_1, ..., i_n}(V_0)$. Note that $\hat{F} = \mathrm{Cl}(\bigcup_{n \geq 0} V_n)$. Then, F ($\hat{F}$) is called a (compact) nested fractal if the following holds in addition to the above conditions:

(NF1) (Symmetry). If $x, y \in V_0$, then reflection in the hyperplane $H_{xy} = \{z: |z - x| = |z - y|\}$ maps V_n to itself.

(NF2) (Nesting). If $\{i_1, ..., i_n\}$, $\{j_1, ..., j_n\}$ are distinct sequences, then

$$\Psi_{i_1, ..., i_n}(\hat{F}) \cap \Psi_{j_1, ..., j_n}(\hat{F}) = \Psi_{i_1, ..., i_n}(V_0) \cap \Psi_{j_1, ..., j_n}(V_0).$$

Line segument is a simple example of the nested fractal, but we will exclude it here. In this paper, we also make the following assumption on nested fractals.

(NF*) There exists $k_0 > 0$ satisfying the following for all $m \geq 0$:

If $x, y \in \hat{F}$ satisfy $\|x - y\| \leq k_0 \alpha^{-m}$, then x, y join either in the same m-complex or adjacent m-complexes.

We remark that we can easily prove that this assumption is equivalent to Assumption 2.2 in [16]. But we do not know whether this holds for all nested fractals or not.

2. *Sierpinski carpets.* Let $n \geqslant 2$, $F_0 = [0, 1]^n$, and let $l \in \mathbf{N}$, $l \geqslant 3$ be fixed. Set $\mathscr{S} = \{\prod_{i=1}^n [(k_i - 1)/l, k_i/l] : k_i \in \{1, ..., l\}\ (\forall i \in \{1, ..., N\})\}$. We assume that each Ψ_i maps F_0 onto some element of $\mathscr{S}$ (thus $\alpha = l$). Set $F_1 = \bigcup_{i=1}^N \Psi_i(F_0)$. We assume $N < l^n$ to exclude the case $F_1 = [0, 1]^n$. Then, F $(\hat{F})$ is called a (compact) Sierpinski carpet if the following holds in addition to the conditions mentioned above:

(SC1) (Symmetry). F_1 is preserved by all the isometries of the unit cube F_0.

(SC2) (Non-diagonality). Let B be a cube in F_0 which is the union of 2^n distinct elements of $\mathscr{S}$. (So B has side length $2l^{-1}$.) Then if $\mathrm{Int}(F_1 \cap B)$ is non-empty, it is connected.

The assumption (SC2) is included by some technical reason which is not essential. Note that the "Borders included" condition, assumed in [5], is not needed in our discussion. Note also that (NF*) always hold for the case of the carpets.

The biggest difference between the two examples is whether the fractal is finitely ramified or not, i.e. whether it can be disconnected by removing a certain finite number of points or not (nested fractals are finitely ramified due to (NF2)). Let $\hat{\mu}$ be a Bernoulli probability measure on $\hat{F}$ such that $\hat{\mu}(\Psi_i(\hat{F})) = (1/N)\ \forall i \in S$. Also, let μ be a Bernoulli measure on F such that $\mu|_{\hat{F}} = \hat{\mu}$. Then, setting $s = \log N / \log \alpha$, we have for each $x \in F$, $r > 0$,

$$c_{4.1} r^s \leqslant \mu(B(x, r) \cap F) \leqslant c_{4.2} r^s,$$

where $c_{4.1}, c_{4.2}$ are positive constants (same estimates hold for $\hat{F}$ by $\hat{\mu}$). Thus μ $(\hat{\mu})$ satisfies (2.9). Note that as $s < n$, we see that $m(F) = 0$ (thus (2.11) is satisfied).

4.2. *Dirichlet Forms on Nested Fractals and Sierpinski Carpets*

We now introduce Dirichlet forms on Sierpinski carpets and nested fractals following [19, 11] and show that their domains satisfy (2.6) for some $\beta \geqslant 1$. Note that Dirichlet forms on nested fractals can be constructed by much more elegant way (see [9, 17]). However, the following construction is more general and both of nested fractals and carpets can be treated by the construction.

For $x, y \in S^n$, we write $x \overset{n}{\sim} y$ if the Hausdorff dimension of the set $F_x(F_0) \cap F_y(F_0)$ equals $d - 1$ for the carpets and if $F_x(F_0) \cap F_y(F_0) \neq \varnothing$ for the nested fractals. Define $q_{xy}^{(n)}$, $x, y \in S^n$ by $q_{xy}^{(n)} = 1$ if $x \overset{n}{\sim} y$ and $q_{xy}^{(n)} = 0$ otherwise. We will first consider the finite graph $(S^n, \{q_{xy}^{(n)}\}_{x, y \in S^n})$. Note that this graph is connected. For a set A, denote $l(A)$ the set of functions on A. Let $\hat{\mathscr{E}}_{(n)}$ be a symmetric bilinear form in $l(S^n)$ defined by

$$\hat{\mathscr{E}}_{(n)}(u, v) = \sum_{x,\, y \in S^n} q^{(n)}_{xy}(u(x) - u(y))(v(x) - v(y)) \qquad u, v \in l(S^n).$$

Now, define a Poincaré constant as

$$\lambda_n = \sup \left\{ \sum_{x \in S^n} (u(x) - \langle u \rangle_{S^n})^2 : u \in l(S^n),\ \hat{\mathscr{E}}_{(n)}(u, u) = 1 \right\},$$

where we denote $\langle u \rangle_B = (1/\sharp B) \sum_{x \in B} u(x)$ for any finite set B and $u \in l(B)$. Let $\{P^{(n)}_x\}_{x \in S^n}$ be a Markov process on S^n which corresponds to $\hat{\mathscr{E}}_{(n)}$. Denoting $T_n = \lambda_n N^n$, let $Q^{(n)}$ be the probability law of $\{\Psi_{w(T_n t)}(0), t \in \mathbf{Q}_+\}$ under $N^{-n} \sum_{x \in S^n} P^{(n)}_x(dw)$ where $\mathbf{Q}_+ \equiv \mathbf{Q} \cap [0, \infty)$. $\{Q^{(n)}\}_{n \geqslant 1}$ are probability measures in $\hat{F}^{\mathbf{Q}_+}$. It can be proved that for each cluster point $\tilde{Q}$ of $\{Q^{(n)}\}$, there is a strongly continuous symmetric Markov semigroup $\{Q_t\}_{t \geqslant 0}$ in $\mathbf{L}^2(\hat{F}, d\hat{\mu})$ whose finite dimensional distributions on rational time coincide with those of $\tilde{Q}$. Denote $\mathscr{D}ch$ be the set of Dirichlet forms associated with the cluster points of $\{Q^{(n)}\}$.

Now, for each $n \geqslant 1$, $\tilde{P}_n: \mathbf{L}^1(\hat{F}, d\hat{\mu}) \to l(S^n)$ and $\iota_n: l(S^n) \to \mathbf{L}^\infty(\hat{F}, d\hat{\mu})$ be given by

$$\tilde{P}_n f(x) = \hat{\mu}(\psi_x(\hat{F}))^{-1} \int_{\psi_x(\hat{F})} f(x)\, \hat{\mu}(dx) \qquad \forall x \in S^n, \quad \forall f \in \mathbf{L}^1(\hat{F}, d\hat{\mu}),$$

$$\iota_n u(y) = u(x), \qquad \text{if} \quad y \in \psi_x(\hat{F}) \qquad \forall x \in S^n, \quad \forall u \in l(S^n).$$

For $n \geqslant 1$, let $\hat{\mathscr{E}}^{(n)}$ be a quadratic form in $\mathbf{L}^2(\hat{F}, d\hat{\mu})$ given by

$$\hat{\mathscr{E}}^{(n)}(f, g) = \lambda_n \hat{\mathscr{E}}_{(n)}(\tilde{P}_n f, \tilde{P}_n g) \qquad \forall f, g \in \mathbf{L}^2(\hat{F}, d\hat{\mu}).$$

Defining $\hat{\mathscr{F}} = \{f : \sup_n \hat{\mathscr{E}}^{(n)}(f, f) < \infty\}$, the following holds (see [5, 11, 19] for the proof).

LEMMA 4.1. (1) *For any $\hat{\mathscr{E}} \in \mathscr{D}ch$, it holds that $\mathscr{D}(\hat{\mathscr{E}}) = \hat{\mathscr{F}}$.*

(2) *There exist $c_{4.3}, c_{4.4} > 0$ such that*

$$c_{4.3} \sup_n \hat{\mathscr{E}}^{(n)}(f, f) \leqslant \hat{\mathscr{E}}(f, f) \leqslant c_{4.4} \liminf_{n \to \infty} \hat{\mathscr{E}}^{(n)}(f, f), \tag{4.1}$$

for any $\hat{\mathscr{E}} \in \mathscr{D}ch$ and $f \in \hat{\mathscr{F}}$.

(3) *There exist $c_{4.5}, c_{4.6} > 0$ and $\lambda > 0$ such that*

$$c_{4.5} \lambda^n \leqslant \lambda_n \leqslant c_{4.6} \lambda^n \qquad \forall n \in \mathbf{N}. \tag{4.2}$$

We can further prove the following (see [11, 19] for the proof).

PROPOSITION 4.2. *$(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ is a local regular Dirichlet form on $\mathbf{L}^2(\hat{F}, d\hat{\mu})$, which satisfies* (2.12).

Note that this form might not have the following scaling property

$$\hat{\mathscr{E}}(f, g) = \lambda \sum_{i=1}^{N} \hat{\mathscr{E}}(f \circ \Psi_i, g \circ \Psi_i), \qquad \forall f, g \in \hat{\mathscr{F}}, \tag{4.3}$$

as we do not know the uniqueness of the cluster point. But using the averaging method in [19], one can reconstruct the Dirichlet form with the same domain, comparable with $\hat{\mathscr{E}}$, and satisfies (4.3) with λ appeared in (4.2). For the case of nested fractals, it is proved that $\lambda > 1$ and $\hat{\mathscr{F}} \subset C(\hat{F})$, the set of continuous functions on $\hat{F}$, but for the carpets, this does not hold when n and N are large. Define $t_F = \lambda N$, $d_w = \log t_F / \log \alpha$ and set $\beta = d_w/2$. It is proved that $t_F \geqslant \alpha^2$ ($t_F > 2$, which will be used later, is also known) so that $\beta \geqslant 1$. As $s < n$, (2.5) is satisfied. We note that $\{X_t\}$, the corresponding diffusion on $\hat{F}$, satisfies $E^x \|X_t - x\| \asymp t^{1/d_w}$ $0 < \forall t < 1$ and it is "diffusive" or "sub-diffusive" as $d_w \geqslant 2$.

We now characterize the domain of the Dirichlet form. For the case of nested fractals, such a characterization is done by [22] as a natural extension of the result of [13] for the case of the Sierpinski gasket. Their proofs rely on the fact $\hat{\mathscr{F}} \subset C(\hat{F})$, but in our case, the element of $\hat{\mathscr{F}}$ is not necessarily continuous (as we include the case of higher dimensional carpets). Our proof is thus an extension of those of [13, 22].

THEOREM 4.3. $\hat{\mathscr{F}} = Lip(\beta, 2\infty)(\hat{F})$.

Proof of $\hat{\mathscr{F}} \subset Lip(\beta, 2\infty)(\hat{F})$. This part of the proof is essentially the same as [22, 13]. Let $f \in \hat{\mathscr{F}} \cap C_0(\hat{F})$. It is enough to show

$$\alpha^{\nu(2\beta + s)} \iint_{\|x-y\| < c_0 \alpha^{-\nu}} |f(x) - f(y)|^2 \, d\hat{\mu}(x) \, d\hat{\mu}(y) < c\hat{\mathscr{E}}(f, f), \tag{4.4}$$

for all $\nu \in \mathbf{N} \cup \{0\}$ with some $c > 0$, as $\hat{\mathscr{F}} \cap C_0(\hat{F})$ is a core of the form. Consider $f_n \equiv \tilde{P}_n f$ as an element of $\mathbf{L}^\infty(\hat{F}, d\hat{\mu})$ which is constant on each interior of n-complexes. Then, as f_n converges uniformly to f as $n \to \infty$ due to $f \in C_0(\hat{F})$, it is enough to show

$$\alpha^{\nu(2\beta + s)} \iint_{\|x-y\| < c_0 \alpha^{-\nu}} |f_n(x) - f_n(y)|^2 \, d\hat{\mu}(x) \, d\hat{\mu}(y) < c\hat{\mathscr{E}}(f, f), \tag{4.5}$$

for all $n > \nu$ with some $c > 0$.

For $x \in S^n$, denote $N_n(x)$ the set of $w \in S^n$ such that $\Psi_x(\hat{F}) \cap \Psi_w(\hat{F}) \neq \emptyset$. Note that $\sharp N_n(x) \leqslant M$ for some $M > 0$ independent of x and n. By our assumption (NF*), if $x \in \Psi_s(\hat{F})$ for some $s \in S^\nu$ and $\|x - y\| \leqslant k_0 \alpha^{-\nu}$, then $y \in \Psi_{s^*}(\hat{F})$ for some $s^* \in N_\nu(s)$. Thus, taking $c_0 = k_0$,

$$\int_{\|x-y\| < c_0 \alpha^{-\nu}} |f_n(x) - f_n(y)|^2 \, d\hat{\mu}(x) \, d\hat{\mu}(y)$$

$$\leqslant N^{-2n} \sum_{s \in S^\nu} \sum_{s^* \in N_\nu(s)} \sum_{x \in S^{n-\nu}} \sum_{y \in S^{n-\nu}} (f_n(s \cdot x) - f_n(s^* \cdot y))^2,$$

where we denote $x \cdot y = x_1 \cdots x_n y_1 \cdots y_m \in S^{n+m}$ for each $x = x_1 \cdots x_n \in S^n$ and $y = y_1 \cdots y_m \in S^m$. Denoting the left hand side of the above inequality by I_ν, we have

$$I_\nu \leqslant 2N^{-2n} \sum_{s \in S^\nu} \sum_{s^* \in N_\nu(s)} \sum_{x \in S^{n-\nu}} \sum_{y \in S^{n-\nu}} \{(f_n(s \cdot x) - f_\nu(s))^2$$

$$+ (f_\nu(s) - f_\nu(s^*))^2 + (f_n(s^* \cdot y) - f_\nu(s^*))^2\}$$

$$\leqslant cN^{-2n} \sum_{s \in S^\nu} \sum_{x \in S^{n-\nu}} \{(f_n(s \cdot x) - f_\nu(s))^2 \, \sharp(S^{n-\nu})\}$$

$$+ cN^{-2n} \sum_{s \in S^\nu} \sum_{s^* \in N_\nu(s)} \{(f_\nu(s) - f_\nu(s^*))^2 \, (\sharp(S^{n-\nu}))^2\}$$

$$\leqslant cN^{-\nu-n} \sum_{s \in S^\nu} \sum_{x \in S^{n-\nu}} (f_n(s \cdot x) - f_\nu(s))^2$$

$$+ cN^{-2\nu} \sum_{s \in S^\nu} \sum_{s^* \in N_\nu(s)} (f_\nu(s) - f_\nu(s^*))^2 \equiv J_\nu^1 + J_\nu^2,$$

for some $c > 0$. We first estimate J_ν^1. Set $s_k = s \cdot x_1 \cdots x_k \in S^{\nu+k}$ for $0 \leqslant k \leqslant n - \nu$. Then $s_0 = s$ and $s_{n-\nu} = s \cdot x$. Using the inequality $(a+b)^2 \leqslant 2(a^2 + b^2)$ repeatedly, we obtain

$$(f_n(s \cdot x) - f_\nu(s))^2 \leqslant \sum_{l=0}^{n-\nu-1} 2^{l+1} (f_{\nu+l}(s_l) - f_{\nu+l+1}(s_{l+1}))^2.$$

Next we perform the summation of this inequality over $x \in S^{n-\nu}$, fixing $s \in S^\nu$. Then, a fixed pair (s_l, s_{l+1}) of the above form appears in the term of the summand whenever $s \cdot x$ coinsides s_{l+1} up to the $(\nu + l + 1)$-th place, thus $N^{n-(\nu+l+1)}$-times. We thus have

$$J_v^1 \leqslant cN^{-v-n} \sum_{s \in S^v} \sum_{l=0}^{n-v-1} \sum_{x \in S^l} 2^{l+1} N^{n-(v+l+1)}$$
$$\times \sum_{j=1}^{N} (f_{v+l}(s \cdot x) - f_{v+l+1}(s \cdot x \cdot j))^2.$$

Now, because $f_{v+l}(s \cdot x) = (1/N) \sum_{k=1}^{N} f_{v+l+1}(s \cdot x \cdot k)$, and each two complexes $\Psi_{s \cdot x \cdot k}(\hat{F})$, $\Psi_{s \cdot x \cdot j}(\hat{F})$ can be connected by at most N-th $(v+l+1)$-complexes for all $k, j \in S$, we have

$$\sum_{s \in S^v} \sum_{x \in S^l} \sum_{j=1}^{N} (f_{v+l}(s \cdot x) - f_{v+l+1}(s \cdot x \cdot j))^2 \leqslant c' \lambda^{-(v+l+1)} \hat{\mathscr{E}}^{(v+l+1)}(f, f),$$

for some $c' > 0$. It is also clear that

$$J_v^2 \leqslant c'' N^{-2v} \lambda^{-v} \hat{\mathscr{E}}^{(v)}(f, f).$$

Noting that $\hat{\mathscr{E}}^{(v+l+1)}(f, f) \leqslant (1/c_{4.3})\, \hat{\mathscr{E}}(f, f)$ (by Lemma 4.1(2)), $\lambda = t_F/N$ and $t_F > 2$, we have

$$\begin{aligned} I_v &\leqslant cc' N^{-n-v} \sum_{l=0}^{n-v-1} 2^{l+1} N^{n-(v+l+1)} (t_F/N)^{-(v+l+1)} \hat{\mathscr{E}}^{(v+l+1)}(f, f) \\ &\quad + c'' N^{-2v} (t_F/N)^{-v}\, \hat{\mathscr{E}}^{(v)}(f, f) \\ &\leqslant c''' (N t_F)^{-v}\, \hat{\mathscr{E}}(f, f), \end{aligned}$$

for all $n > v$. As $2\beta + s = \log(t_F N)/\log \alpha$,

$$\alpha^{v(2\beta+s)} I_v \leqslant c''' \hat{\mathscr{E}}(f, f),$$

and (4.5) is proved. ∎

Proof of $\hat{\mathscr{F}} \supset Lip(\beta, 2\infty)(\hat{F})$. Although we do not use this part of the theorem in this paper, we will write down the proof as the proof is quite simple compared to those of [13] and [22]. Let $f \in Lip(\beta, 2\infty)(\hat{F})$ and consider $f_n \equiv \tilde{P}_n f$ as an element of $\mathbf{L}^\infty(\hat{F}, d\hat{\mu})$ as before. We will first show

$$\begin{aligned} &c_1 N^{-2n} \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} (f_n(s) - f_n(s^*))^2 \\ &\qquad \leqslant \iint_{\|x-y\| \leqslant c_0 \alpha^{-n}} |f(x) - f(y)|^2 \, d\hat{\mu}(x)\, d\hat{\mu}(y), \end{aligned} \tag{4.6}$$

for some constants $c_0, c_1 > 0$. Indeed, taking c_0 large enough, the right hand side is greater than or equal to

$$\sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} \int_s \int_{s^*} |f(x) - f(y)|^2 \, d\hat{\mu}(x) \, d\hat{\mu}(y)$$

$$= \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} \left\{ N^{-n} \int_s f(x)^2 \, \hat{\mu}(x) \right.$$

$$\left. + N^{-n} \int_{s^*} f(y)^2 \, \hat{\mu}(y) - 2N^{-2n} f_n(s) \, f_n(s^*) \right\}$$

$$= N^{-n} \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} \left\{ \int_s |f(x) - f_n(s)|^2 \, d\hat{\mu}(x) + \int_{s^*} |f(y) - f_n(s^*)|^2 \, d\hat{\mu}(y) \right\}$$

$$+ N^{-2n} \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} (f_n(s) - f_n(s^*))^2,$$

as $\hat{\mu}(s) = \hat{\mu}(s^*) = N^{-n}$. We thus obtain the inequality.

Now, by (4.6) and the fact $f \in \mathrm{Lip}(\beta, 2\infty)(\hat{F})$,

$$N^{-2n} \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} (f_n(s) - f_n(s^*))^2 \leqslant M\alpha^{-n(2\beta + s)},$$

for some $M > 0$. Using this and the fact $\lambda = t_F / N$, $\alpha^{2\beta + s} = t_F N$, we have

$$\hat{\mathscr{E}}^{(n)}(f, f) = \lambda^n \hat{\mathscr{E}}_{(n)}(f_n, f_n) \leqslant \lambda_n \sum_{\substack{s \in S^n \\ s^* \in N_n(s)}} (f_n(s) - f_n(s^*))^2$$

$$\leqslant M(t_F N)^n \, \alpha^{-n(2\beta + s)} \leqslant M.$$

As $M > 0$ is independent of n, we obtain $f \in \hat{\mathscr{F}}$ due to (4.1). ∎

Remark 4.4. (1) In the proof of [13] and [22], they relied on the $(d_w - d_f)$-Hölder continuity of the Lipschitz space and the fact $d_w - d_f > 0$, but they are needless as we see. In fact, their proof can be applied for the case $g \in Lip(\beta, 2\infty)(\hat{F}) \cap C_0(\hat{F})$ by the following modifications (here we use the notation of [22]).

- Instead of using Hölder continuity, use the fact

$$\frac{1}{\hat{\mu}(T_\nu)} \int_{T_\nu} (g(x) - g(p_\nu))^2 \, d\nu(p_\nu) \to 0 \qquad as \quad \nu \to \infty.$$

- Take v to be an odd number and take $T_{2i} = S_0$, $T_{2i+1} = S_{2i+1}$, $k = 1$. (Then, we can save M^r and the sequence converges when $t_F > 2$, which is true for nested fractals and carpets.)

(2) As $\hat{\mu}$ satisfies (2.9), $f \in \hat{\mathscr{F}} = \mathrm{Lip}(\beta, 2, \infty)(\hat{F})$ is bounded and $(\beta - s/2)$-Hölder continuous if $0 < \beta - s/2 < 1$, i.e. there is a constant $M > 0$ such that $|f(x)| \leqslant M$ and $|f(x) - f(y)| \leqslant M\,|x - y|^{\beta - s/2}$ $\forall x, y \in F$, $\|x - y\| \leqslant 1$ (see [13, Corollary 2]). This holds for nested fractals, but as mentioned before, $f \in \hat{\mathscr{F}}$ is not even continuous in general for the carpets.

We now define Dirichlet forms on F, using the form $(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ on $\hat{F}$ which satisfies (4.3). Set $\hat{F}_{\langle l \rangle} = \alpha^l \hat{F}$ and define σ_l: $l(\hat{F}_{\langle l \rangle}) \to l(\hat{F})$ by $\sigma_l f(x) = f(\alpha_1^l x) = f \circ \Psi_1^{(-l)}(x)$ $\forall x \in \hat{F}$. Set $\hat{\mathscr{F}}_{\hat{F}_{\langle l \rangle}} = \sigma_{-l} \hat{\mathscr{F}}$ and $\hat{\mathscr{E}}_{\hat{F}_{\langle l \rangle}}(f, g) = \lambda^{-l} \hat{\mathscr{E}}(\sigma_l f, \sigma_l g)$ $\forall f, g \in \hat{\mathscr{F}}_{\mathscr{F}_{\langle l \rangle}}$. It is easy to see

$$\hat{\mathscr{E}}_{\hat{F}_{\langle l-1 \rangle}}(f|_{\hat{F}_{\langle l-1 \rangle}}, f|_{\hat{F}_{\langle l-1 \rangle}}) \leqslant \hat{\mathscr{E}}_{\hat{F}_{\langle l \rangle}}(f, f) \qquad \forall f \in \hat{\mathscr{F}}_{\mathscr{F}_{\langle l \rangle}}. \tag{4.7}$$

Now define

$$\mathscr{D}_F = \{ f \in C_0(F) : f|_{\hat{F}_{\langle l \rangle}} \in \mathscr{F}_{\hat{F}_{\langle l \rangle}} \ \forall l \in \mathbf{N}, \lim_{l \to \infty} \hat{\mathscr{E}}_{\hat{F}_{\langle l \rangle}}(f|_{\hat{F}_{\langle l \rangle}}, f|_{\hat{F}_{\langle l \rangle}}) < \infty \},$$

$$\mathscr{E}(f, g) = \lim_{l \to \infty} \hat{\mathscr{E}}_{\hat{F}_{\langle l \rangle}}(f|_{\hat{F}_{\langle l \rangle}}, g|_{\hat{F}_{\langle l \rangle}}) \qquad \forall f, g \in \mathscr{D}_F.$$

It is easy to show that $(\mathscr{E}, \mathscr{D}_F)$ is closable in $\mathbf{L}^2(F, d\mu)$ by using (4.7). Denote $\mathscr{F} = \overline{\mathscr{D}_F}^{\mathscr{E}_1}$ so that $(\mathscr{E}, \mathscr{F})$ is the smallest extension of $(\mathscr{E}, \mathscr{D}_F)$. Then we have the following.

PROPOSITION 4.5. *$(\mathscr{E}, \mathscr{F})$ is a local regular Dirichlet form on $\mathbf{L}^2(F, \mu)$ which satisfies* (2.12), (2.14) *and the following scaling property*,

$$\mathscr{E}(f, g) = \lambda \mathscr{E}(f \circ \Psi_1, g \circ \Psi_1) \qquad \forall f, g \in \mathscr{F}.$$

Indeed, the key is to show the regularity, but this is also easy by using Stone–Weierstrass' theorem in the same way as the proof of Theorem 2.3. See [5, 16] for the proof of (2.12), (2.14).

We now prove the following.

THEOREM 4.6. $\mathscr{F} \subset Lip(\beta, 2\infty)(F)$.

Proof. Noting that

$$\int_{\hat{F}} \sigma_1 f(x)\, d\mu(x) = \frac{1}{N} \int_{\hat{F}_{\langle 1 \rangle}} f(x')\, d\mu(x'),$$

we have for each $f \in \hat{\mathscr{F}}$ and $\nu, l \in \mathbf{N} \cup \{0\}$,

$$\alpha^{\nu(2\beta+s)} \iint_{\|x-y\|<c_0\alpha^{-\nu}} |\sigma_l f(x) - \sigma_l f(y)|^2 \, d\hat{\mu}(x)\, d\hat{\mu}(y)$$

$$= \lambda^l \alpha^{(\nu-l)(2\beta+s)} \iint_{\substack{x,\, y \in \hat{F}_{\langle l \rangle} \\ \|x-y\|<c_0\alpha^{l-\nu}}} |f(x)-f(y)|^2 \, d\mu(x)\, d\mu(y). \tag{4.8}$$

By this and (4.4), we have

$$\alpha^{(\nu-l)(2\beta+s)} \iint_{\substack{x,\, y \in \hat{F}_{\langle l \rangle} \\ \|x-y\|<c_0\alpha^{l-\nu}}} |f(x)-f(y)|^2 \, d\hat{\mu}(x)\, d\hat{\mu}(y) < c\hat{\mathscr{E}}_{\hat{F}_{\langle l \rangle}}(f, f),$$

for all $f \in \hat{\mathscr{F}}_{\hat{F}_{\langle l \rangle}}$ and $\nu, l \in \mathbf{N} \cup \{0\}$. Taking $\nu' = \nu - l$, we have

$$\alpha^{\nu'(2\beta+s)} \iint_{\|x-y\|<c_0\alpha^{-\nu'}} |f(x)-f(y)|^2 \, d\mu(x)\, d\mu(y) < c\mathscr{E}(f, f),$$

for all $f \in \mathscr{F}$, $\nu' \in \mathbf{N} \cup \{0\}$ by the monotone convergence theorem, and the proof is completed. ∎

Note that by the proof of Theorem 4.3 and by (4.8), we can prove that

$$\mathscr{F} = \widetilde{\mathrm{Lip}}(\beta, 2\infty)(F),$$

where $\widetilde{\mathrm{Lip}}(\beta, 2, \infty)(F)(\subset \mathrm{Lip}(\beta, 2\infty)(F))$ is a set of $f \in \mathbf{L}^2(F, d\mu)$ such that $\sup_{\nu \in \mathbf{Z}} \alpha^{\nu(2\beta+s)} \iint_{\|x-y\|<c_0\alpha^{-\nu}} |f(x)-f(y)|^2 \, d\mu(x)\, d\mu(y) < \infty$.

Finally, we mention that when $s > n-2$, we see that $\tilde{X}$ penetrates into F ($\hat{F}$) by Theorem 2.9. Further, as the diffusion on F satisfies (2.14), we have the heat kernel estimate (2.15) by Theorem 2.11.

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